5.6: Substitution Methods (Definite Integrals)

Theorem:

$$\int_a^b f(g(x))\cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du \quad (\text{where } u=g(x))$$

Ex(1): $\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx$ Let $u=x^3+1$. Then $du=3x^2dx$

Method 1: Changing Limits of Integration

Letting $u=g(x)$, we have $g(1)=1^3+1=2$ and $g(-1)=(-1)^3+1=0$

So $\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx = \int_0^2 \sqrt{u}\,du = \frac{u^{3/2}}{3/2}\Big|_0^2 = \frac{2}{3}\cdot 2^{3/2}.$

Method 2: Not changing Limits, but evaluating x instead of u.

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx = \int_{\square}^{\square}\sqrt{u}\,du = \frac{u^{3/2}}{3/2}\Big|_{\square}^{\square} = \frac{2(x^3+1)^{3/2}}{3}\Big|_{-1}^{1} = \frac{2}{3}\cdot 2^{3/2}$$

Ex(2): $\int_{\pi/4}^{\pi/2} \cot\theta\csc^2\theta\,d\theta$ $u=\csc\theta$ then $du=-\cot\theta\csc\theta\,d\theta$.

So $\csc\frac{\pi}{4}=\sqrt{2}$, $\csc(\frac{\pi}{2})=1$ and

$$\int_{\sqrt{2}}^{1} -u\,du = \int_1^{\sqrt{2}} u\,du = \frac{u^2}{2}\Big|_1^{\sqrt{2}} = 1-\frac{1}{2}=\frac{1}{2}.$$

Theorem: (a) If f is even, then

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

(b) If f is odd, then

$$\int_{-a}^{a} f(x)\,dx = 0.$$

Ex(3): $\int_{-2}^{2} x^4 - 4x^2 + 6\,dx = 2\int_{0}^{2} x^4 - 4x^2 + 6\,dx$

$$= 2\left(\frac{x^5}{5} - \frac{4x^3}{3} + 6x\right)\Big|_0^2 = 2\left(\frac{32}{5} - \frac{32}{3} + 12\right) = \frac{232}{15}.$$